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EXPLICIT SOLUTIONS OF MULTIDIMENSIONAL INVERSE UNSTEADY HEAT-CONDUCTION PROBLEMS

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Explicit solutions are found of a number of inverse problems of determining the thermal conductivity in linear and nonlinear heat transport.

The determination of variable thermophysical characteristics of media is one of the urgent problems of contemporary thermophysics. Recently there has been a rapid development of the theory of multidimensional inverse problems [1-5]. In these investigations great importance is attached to the development of special methods which yield explicit solutions. These solutions can serve directly as a basis for experimental methods of determining variable physical characteristics of media.

We consider a thermal process described by the system

$$C(x,\ t)\,T_t - \nabla\lambda(x,\ t)\,\nabla T + \alpha(x,\ t)\,T = Q(x,\ y,\ t), \tag{1}$$

$$T|_{t=0} = \varphi(x,\ y), \tag{2}$$

$$T|_{\bar{D}_1 X \Gamma_2} = 0,\ T|_{\Gamma_1 X \bar{D}_2} = f(\xi,\ y,\ t). \tag{3}$$

If the quantities C, λ, α, Q, φ, and f are known, system (1)-(3) can be used to calculate the temperature distribution T(x, y, t). Our primary problem is to determine the thermal conductivity λ(x, t). To do this we supplement system (1)-(3) by the condition

$$\left.\frac{\partial T}{\partial \nu}\right|_{y=\eta} = \gamma(x,\ t), \tag{4}$$

which is the expression for the temperature gradient on the plane y = η, where η is a fixed point on the boundary Γ_2. The coefficient λ(x, t) is sought in the class of continuous and positive functions.

Questions of the correctness of problems of the type (1)-(4) were studied in [4]. We consider cases for which the solutions can be found in explicit form.

We denote by ω(y) the normalized eigenfunction of the operator $-\Delta_y$ corresponding to the eigenvalue $\mu > 0$, i.e.,

$$-\Delta_y \omega(y) = \mu\omega(y),\ \omega(y)|_{\Gamma_2} = 0,\ y \in D_2. \tag{5}$$

If m = 1, $\bar{D}_2 \equiv [0,\ 1]$, then $\omega(y) = \sin k\pi y$, $\mu = k^2\pi^2$, where k is a positive integer. It is not difficult to indicate the general form of the function ω(y) for a number of other domains also.

We consider a thermal process in which the following conditions are realized:

a) $Q(x,\ y,\ t) = Q_0(x,\ t)\omega(y)$, $\varphi(x,\ y) = \varphi_0(x)\omega(y)$, $f(\xi,\ y,\ t) = f_0 \times (\xi,\ t)\ \omega(y)$, where $Q_0(x,\ t)$, $\varphi_0(x)$, $f_0(\xi,\ t)$ are given functions;

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b) $\gamma(\xi, t) = f_0(\xi, t)\ \partial\omega/\partial\nu(\eta),\ \gamma(x, 0) = \varphi_0(x)\ \partial\omega/\partial\nu(\eta),\ \xi \in \Gamma_1,\ \partial\omega/\partial\nu(\eta) \neq 0.$

This condition is an expression of the compatibility of the input data of the problem, and it must be satisfied in actual processes.

We multiply Eq. (1) by $\omega(y)$ and integrate over domain D_2.

Using the notation

$$T_0(x, t) = \int_{D_2} T(x, y, t)\,\omega(y)\,dy, \tag{6}$$

then

$$C(x, t)\,T_{0t} - \nabla_x\lambda(x, t)\,\nabla_x T_0 + [\mu\lambda(x, t) + \alpha(x, t)]\,T_0 = Q_0(x, t). \tag{7}$$

After substituting (6), conditions (2)-(4) take the form

$$T_0|_{t=0} = \varphi_0(x),\quad T_0|_{\Gamma_2} = f(\xi, t), \tag{8}$$

$$T_0 \frac{\partial\omega}{\partial\nu}\Big|_{y=\eta} = \gamma(x, t)\ . \tag{9}$$

It follows from (9) that

$$T_0\ (x, t) = \gamma(x, t)\left[\frac{\partial\omega}{\partial\nu}(\eta)\right]^{-1}. \tag{10}$$

If we substitute this expression into (7), we obtain

$$-\nabla_x\lambda(x, t)\,\nabla_x\gamma + \mu\lambda(x, t)\,\gamma = \Phi(x, t), \tag{11}$$

where

$$\Phi(x, t) = Q_0(x, t)\frac{\partial\omega}{\partial\nu}(\eta) - \alpha(x, t)\,\gamma(x, t) - C(x, t)\,\gamma_t(x, t).$$

Thus, we obtain a first order partial differential equation for the function $\lambda(x, t)$. We seek the solution of this equation for $\bar{D}_1 \equiv [0, 1]$. Then (11) takes the form

$$-\lambda_x(x, t)\,\gamma_x(x, t) + \lambda(x, t)\,[\mu\gamma(x, t) - \gamma_{xx}(x, t)] = \Phi(x, t). \tag{12}$$

Hence it follows that the function $\lambda(x, t)$ is uniquely determined only if its value is given at one point of the interval [0, 1]. We assume that the function $\gamma_x(x, t)$ vanishes only at the point $x_0 \in [0, 1]$. Then we obtain from (12)

$$\lambda(x_0, t) = \Phi(x_0, t)\,[\mu\gamma(x_0, t) - \gamma_{xx}(x_0, t)]^{-1}. \tag{13}$$

The solution of Eq. (12) which satisfies (13) has the form [6]

$$\lambda(x, t) = \exp\left\{-\int_{x_0}^{x} P(z)\,dz\right\}\left[\int_{x_0}^{x} R(z)\exp\left\{\int_{x_0}^{z} P(\zeta)\,d\zeta\right\}dz + \lambda(x_0, t)\right], \tag{14}$$

where

$$P(z) = [\gamma_{zz}(z, t) - \mu\gamma(z, t)]\,[\gamma_z(z, t)]^{-1};$$
$$R(z) = \Phi(z, t)\,[\gamma_z(z, t)]^{-1};\ z \neq x_0.$$

The right-hand sides of Eqs. (13) and (14) are assumed positive, continuous, and finite.

In practice it is sometimes more convenient to replace (4) in problem (1)-(4) by the condition

$$\lambda\frac{\partial T}{\partial\nu}\Big|_{y=\eta} = q(x, t), \tag{15}$$

i.e., the heat flux density through the plane $y = \eta$.

We now consider the problem of determining the function $\lambda(x, t)$ from conditions (1)-(3) and (15). After the substitution of (6) this system is transformed into the form of (7) and (8):

$$\lambda T_0 \frac{\partial \omega}{\partial \nu}\Big|_{y=\eta} = q(x, t). \tag{16}$$

We have from (7) and (16)

$$C(x, t) T_{0t} - \nabla_x\left[q(x, t) \left(T_0 \frac{\partial \omega}{\partial \nu}(\eta)\right)^{-1} \nabla_x T_0 \right] + \alpha(x, t) T_0 = Q_0(x, t) - \mu q(x, t)\left[\frac{\partial \omega}{\partial \nu}(\eta)\right]^{-1}. \tag{17}$$

Consequently, the function $T_0(x, t)$ is the solution of the mixed problem for the quasilinear parabolic equation (17) with conditions (8) and (9). We assume that $q(x, t) > 0$, $d\omega/d\nu\ (\eta) > 0$, $\varphi_0(x) > 0$, and $f_0\ (\xi, t) > 0$. These assumptions can be ensured experimentally, and some of these conditions are also related to the compatibility of the input data of the problem. Then the function $T_0(x, t) > 0$, and from (17), (8), and (9) it can be found exactly or approximately. Substituting the expression found for $T_0(x, t)$ into (16), we obtain

$$\lambda(x, t) = q(x, t)\left[T_0(x, t) \frac{\partial \omega}{\partial \nu}(\eta)\right]^{-1}.$$

If the coefficient $\lambda(x, t)$ is given, the coefficient $C(x, t)$ or $\alpha(x, t)$ can be found from system (1)-(4) or from (1)-(3) and (15).

We present a special case of the inverse problem considered above. Suppose it is required to determine the function $\alpha(t)$ which is continuous in $[0, t_{ob}]$ and satisfies the equation

$$T_t - \Delta T + \alpha(t) T = 0,\ (x, t) \in \Omega_1, \tag{18}$$

the initial and boundary conditions

$$T|_{t=0} = \varphi(x),\ T|_{\Gamma_1} = 0 \tag{19}$$

and the following supplementary condition

$$\frac{\partial T}{\partial \nu}(\eta, t) = \gamma(t),\ 0 \leqslant t \leqslant t_{ob}, \tag{20}$$

where $\varphi(x)$ and $\gamma(t)$ are given functions. This problem is encountered in the study of the cooling of a body by a stream of liquid or gas, varying the velocity or temperature.

Let D_1 be a domain such that the Green's function $G_1(x, t; \xi, \theta)$ of the first boundary value problem for the equation $T_t - \Delta T = 0$ can be found in explicit form. For example, D_1 can be a half space, a sphere, a segment, etc. We make the substitution

$$v(x, t) = T(x, t) \exp\left\{ \int_0^t \alpha(\tau)\, d\tau \right\} \tag{21}$$

into Eqs. (18)-(20), and after some simple transformations we obtain the following expression for the unknown coefficient $\alpha(t)$:

$$\alpha(t) = \frac{d}{dt} \ln \frac{1}{\gamma(t)} \int_{D_1} \frac{\partial G_1}{\partial \nu}(\eta, t; \xi, 0)\, \varphi(\xi)\, d\xi. \tag{22}$$

If in problem (18)-(20) we specify the condition $(\partial T/\partial\nu)|_{\Gamma_1} = 0$ instead of the condition $T|_{\Gamma_2} = 0$, then instead of (20) giving the condition

$$T(\eta, t) = f(t) \tag{23}$$

we find for the unknown coefficient $\alpha(t)$ the expression

$$\alpha(t) = \frac{d}{dt} \ln \frac{1}{f(t)} \int_{D_1} G_2(\eta, t; \xi, 0)\, \varphi(\xi)\, d\xi, \tag{24}$$

TABLE 1. Comparison of Exact and Approximate Values of the Thermal Conductivity

λ	x										
	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\bar{\lambda}(x;\ 0{,}5)$	1,5	1,6	1,7	1,8	1,9	2	2,1	2,2	2,3	2,4	2,5
$\tilde{\lambda}(x;\ 0{,}5)$	1,5	1,558	1,684	1,787	1,891	2,006	2,105	2,200	2,296	2,402	2,5

where $G_2(x,\ t;\ \xi,\ \theta)$ is the Green's function of the second boundary value problem for the equation $T_t - \Delta T = 0$.

In conclusion, we consider the inverse problem for quasilinear heat-conduction equations. In this problem we obtain an explicit expression for the coefficients being sought by using self-similar solutions of the equation under consideration.

Suppose it is required to find the thermal conductivity $\lambda(T) > 0$ when the following conditions are satisfied:

$$C(T)\,T_t - x^{-k}[x^k\lambda(T)\,T_x]_x = 0,\ 0<x<\infty,\ 0<t\leqslant t_{ob}, \tag{25}$$

$$T|_{t=0} = 0,\ T|_{x=0} = f,\ 0\leqslant x<\infty,\ 0<t\leqslant t_{ob}, \tag{26}$$

$$T|_{x=\eta} = \psi(t),\ \psi(0) = 0,\ \psi(\infty) = f, \tag{27}$$

Where $C(T) > 0$ and $\Psi(t)$ are given functions; $\eta > 0$, $f > 0$, $k \geq 0$ are given numbers. In this problem $\lambda(T)$ is sought in the class of positive functions which are continuous for $T\in[0,\ f]$. For $k = 0$ Eq. (25) describes the propagation of heat in a one-dimensional rod, and $k = 1$ and $k = 2$ correspond to spherical and cylindrical symmetries.

Equation (25) for conditions (26) admits a self-similar solution [7], and from the maximum principle there follows the estimate: $0 \leq T(x,\ t) \leq f$. It is easy to confirm that Eq. (25) and conditions (26) remain unchanged for the following transformation of independent variables: $x' = mx$, $t' = m^2t$. Therefore, the function $T(x,\ t)$ must satisfy the identity

$$T(x,\ t) = T(mx,\ m^2t).$$

Setting $m = (t\eta^2)^{-1/2}$, we obtain

$$T(x,\ t) = T\left(\frac{x}{\eta\sqrt{t}},\ \frac{1}{\eta^2}\right) = w\left(\frac{x}{\eta\sqrt{t}}\right).$$

Thus, $T(x,\ t)$ depends only on the argument $x/\eta\sqrt{t}$.

With the notation $z = x/\eta\sqrt{t}$, system (25)-(27) takes the form

$$z^{-k}[z^k\lambda(w)\,w_z]_z = -\frac{1}{2}z\eta^2C(w)\,w_z,\ 0<z<\infty, \tag{28}$$

$$w(0) = f,\ w(\infty) = 0, \tag{29}$$

$$w(z) = \psi(z^{-2}),\ 0<z<\infty. \tag{30}$$

Suppose the following conditions are satisfied:

1) the expression

$$[\psi_t(t)\,t^{\frac{1}{2}(3-k)}]^{-1}\int\limits_0^t \theta^{-\frac{1}{2}(k+1)}\,C(\psi(\theta))\,d\theta$$

for $\forall t\in[0,\ \infty]$ is a strictly positive, continuous, and bounded function;

2) $\psi(t)$ has the inverse $\Phi(\psi)$, defined on $[0,\ f]$ with a range of values in $[0,\ \infty)$.

Then the expression

$$\lambda(T) = \eta^2\Phi_T(T)\,[\Phi(T)]^{\frac{1}{2}(3-k)}\int\limits_0^T C(v)\,v^{-\frac{k+1}{2}}\,dv. \tag{31}$$

is valid for the function $\lambda(T)$. Here $\lambda(0)$ is understood in the sense of the limit of the right-hand side as $T \to +0$.

By integrating Eq. (28) in the domain (z, ∞) and taking account of boundary condition (29), we obtain

$$z^k \lambda(w) w_z = \frac{1}{2} \eta^2 \int_z^\infty \xi^{k+1} C(w) w_\xi d\xi.$$

Hence

$$\lambda(w) = \eta^2 [2z^k w_z]^{-1} \int_z^\infty \xi^{k+1} C(w) w_\xi d\xi.$$

Substituting into this the expression for w(z) from Eq. (30), we find

$$\lambda(\psi(z^{-2})) = \eta^2 \left[2z^k \frac{d}{dz} \psi(z^{-2}) \right]^{-1} \int_z^\infty \xi^{k+1} C(\psi(\xi^{-2})) \qquad \frac{d}{d\xi} \psi(\xi^{-2})\, d\xi.$$

By setting $\zeta = z^{-2}$, $\theta = \xi^{-2}$, the last equality can be written in the form

$$\lambda(\psi) = \eta^2 \zeta^{\frac{k-3}{2}} [\psi_\zeta(\zeta)]^{-1} \int_0^\zeta \theta^{-\frac{k+1}{2}} C(\psi(\theta)) \psi_\theta(\theta)\, d\theta. \tag{32}$$

The function $\psi(t)$ has the inverse $\Phi(\psi)$. Therefore, we obtain from (32)

$$\lambda(\psi) = \eta^2 \Phi_\psi(\psi) [\Phi(\psi)]^{\frac{3-k}{2}} \int_0^\psi C(v) v^{-\frac{k+1}{2}} dv.$$

Here ψ takes on all values in the interval [0, f]. Consequently, the validity of Eq. (31) is proven.

The analog of the inverse problem (28)-(30) was solved in [5] by specifying the condition $T(x, t_{ob}) = \psi(x)$ instead of (30).

Numerical calculations were performed on the model examples. We present one of them. Suppose it is required to find the thermal conductivity $\lambda(x, t) > 0$ from the conditions

$$T_t - \nabla\lambda(x, t) \nabla T = -(3 + 2t + 2x) \sin \pi y, \quad 0 < x, \ y, \ t < 1,$$
$$T|_{t=0} = x^2 \sin \pi y, \quad T|_{x=0} = t \sin \pi y, \quad T|_{x=1} = (1 + t) \sin \pi y,$$
$$T|_{y=0} = T|_{y=1} = 0, \quad \lambda T_y|_{y=0} = \pi(1 + x + t)(x^2 + t).$$

Let $\bar{\lambda}(x, t)$ be the exact value of the thermal conductivity and $\tilde{\lambda}(x, t)$ its approximate value found by substituting (17), (8), and (9) into Eq. (16). In this case $\bar{\lambda}(x, t) = 1 + x + t$, $\omega(y) = \sin \pi y$, $\omega_y(0) = \pi$. It follows from (16) that

$$\lambda(x, t) = (1 + x + t)(x^2 + t)[T_0(x, t)]^{-1}. \tag{33}$$

The function $T_0(x, t)$ is the solution of the problem

$$T_{0t} - (q(x, t)(\pi T_0)^{-1} T_{(x)})_x = -(3 + 2x + 2t) - q(x, t),$$
$$T_0|_{t=0} = x^2, \quad T_0|_{x=0} = t, \quad T_0|_{x=1} = 1 + t.$$

Solving this problem by the method of [7], we obtain an approximate value of the function $T_0(x, t)$. If we substitute this value of $T_0(x, t)$ into (33), we find $\tilde{\lambda}_0 x, t)$.

Table 1 lists the exact and approximate values of the thermal conductivity $\lambda(x, t)$ for $t = 0.5$ and at the nodes of the net $\{x\colon x + x_i,\ x_i = ih,\ h = 0,\ i = 0, 1, 2, \ldots, 10\}$.

NOTATION

t, time; t_{ob}, observation time interval; D_1 and D_2, domains in n-dimensional and m-dimensional Euclidean spaces E_n and E_m, respectively; Γ_1 and Γ_2, boundaries; $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_m)$, arbitrary points in domains D_1 and D_2; T, temperature distribution; C, volumetric heat capacity; λ, thermal conductivity; α, heat-transfer coefficient; Q, internal source strength; φ and f, temperature distributions at beginning of process and on boundary of domain respectively; ν, outward normal to boundary Γ_2; γ, temperature gradient; q, heat flux density; η, fixed point on boundary Γ_2; $D \equiv D_1XD_2$; $\Omega \equiv DX(0, t_{ob}]$, $\Omega_1 \equiv D_1X(0, t_{ob}]$.

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FINITE INTEGRAL TRANSFORMS FOR HEAT AND MASS TRANSFER PROBLEMS IN NONSTATIONARY AND INHOMOGENEOUS MEDIA

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UDC 536.2

Several boundary-value problems of heat and mass transfer are solved for equations with varying coefficients.

Integral transforms are widely used in solving transport problems, mostly described by equations with constant coefficients. Important contributions in developing the method of finite integral transforms were made by Grinberg [1], Tranter [2], the authors of [3-6], etc.

In the present study we construct finite integral transforms for several boundary-value problems of heat and mass transfer, described by equations with varying coefficients. Kernels and norms of the transforms and characteristic equations for finding eigenvalues are determined for these problems. In this case it is important to develop an approach to solving these equations, as suggested by the present author [7].

Consider the problem

$$a(t) f(r) \frac{\partial T}{\partial t} = b(t) \frac{1}{r^{\nu}} \frac{\partial}{\partial r}\left[r^{\nu}\lambda(r) \frac{\partial T}{\partial r}\right] + b(t)\varphi(r) \frac{\partial T}{\partial r} + g(r, t) T + W(r, t), \tag{1}$$

$$R_1 < r < R_2, \left[\lambda(r) \frac{\partial T}{\partial r} + \alpha_1 T \right]_{r=R_1} = \beta_1(t),$$
$$\left[\lambda(r) \frac{\partial T}{\partial r} + \alpha_2 T \right]_{r=R_2} = \beta_2(t), \tag{2}$$

where α_1, α_2 are constant, and $\nu = 0, 1, 2$ are shape coefficients of the geometric region. We introduce the notation

$$\int_{R_1}^{R_2} r^{\nu}\Phi(pr) T(r, t)\, dr = \bar{T}(p, t), \tag{3}$$

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